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$\mathrm{Sp}(8)$ invariant higher spin theory, twistors and geometric BRST formulation of unfolded field equations

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Abstract: We discuss twistor-like interpretation of the Sp(8) invariant formulation of $4d$ massless fields in ten dimensional Lagrangian Grassmannian $\mathrm{Sp}(8)/P$ which is the generalized space-time in this framework. The correspondence space $\mathbf{C}$ is $SpH(8)/PH$ where $SpH(8)$ is the semidirect product of Sp(8) with Heisenberg group H_M and PH is some quasiparabolic subgroup of $SpH(8)$. Spaces of functions on $\mathrm{Sp}(8)/P$ and $SpH(8)/PH$ consist of Q_P closed functions on Sp(8) and Q_{PH} closed functions on $SpH(8)$, where Q_P and Q_{PH} are canonical BRST operators of P and PH. The space of functions on the generalized twistor space $\mathbf{T}$ identifies with the $SpH(8)$ Fock module. Although $\mathbf{T}$ cannot be realized as a homogeneous space, we find a nonstandard $SpH(8)$ invariant BRST operator $\mathbf{Q}$ ($\mathbf{Q}^2 = 0$) that gives rise to an appropriate class of functions via the condition $\mathbf{Q}f = 0$ equivalent to the unfolded higher-spin equations. The proposed construction is manifestly Sp(8) invariant, globally defined and coordinate independent. Its Minkowski analogue gives a version of twistor theory with both types of chiral spinors treated on equal footing. The extensions to the higher rank case with several Heisenberg groups and to the complex case are considered. A relation with Riemann theta functions, that are $\mathbf{Q}$-closed, is discussed.

Keywords: Space-Time Symmetries, Global Symmetries, BRST Symmetry

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Contents

JHEP12(2009)021

1 Introduction

In [1] Fronsdal suggested that a set of massless fields of all spins in the four dimensional space should possess a natural description in the Sp(8) invariant ten dimensional Lagrangian Grassmannian Sp$(8|\mathbb{R})/P$ described by the diagram  , that results from crossing out the right node of the Dynkin diagram of Sp(8) thus indicating which parabolic subgroup of Sp(8) should be chosen in the quotient space construction [2]. The big cell of $\mathcal{CM}_M = \mathrm{Sp}(2M|\mathbb{R})/P$ we denote $\mathcal{M}_M$ for any M. Local coordinates of $\mathcal{M}_M$ are real symmetric matrices $X^{AB} = X^{BA}$ $(A, B, \ldots = 1, \ldots M)$. The conclusion that $4d$ massless fields can be described in $\mathcal{M}_4$ was also reached in [3].

The dynamical equations in $\mathcal{M}_M$, that for $M = 4$ are equivalent to the field equations for massless fields of all spins in the four dimensional Minkowski space, were obtained in [4] by using so-called unfolded formulation of the massless field equations [5–8]

$$\left(\frac{\partial}{\partial X^{AB}} + \mu \frac{\partial^2}{\partial Y^A \partial Y^B}\right) C(Y|X) = 0\,, \tag{1.1}$$

where Y^A were treated as auxiliary commuting variables and μ is an arbitrary non-zero parameter introduced for future convenience. It should be stressed that the infinite systems of $4d$ massless fields of all spins $s = 0, 1, 2, \dots$ and $s = 1/2, 3/2, \dots$ described by (1.1) are particularly interesting as constituting fundamental multiplets of fields that appear in the $4d$ nonlinear theories of higher-spin massless fields (see [9] and references therein).

The equations (1.1) express the first derivatives with respect to space-time coordinates X^{AB} in terms of the fields themselves. As such, they belong to the class of *unfolded* partial differential equations that, more generally, express the exterior differential of a set of differential forms in terms of exterior products of the differential forms themselves (for more detail see section 8). Such a first-order form of dynamical field equations can always be achieved by introducing a (may be infinite) set of auxiliary fields which parametrize all combinations of derivatives of the dynamical fields that remain unrestricted by the field equations.

For example, in the system (1.1), the *dynamical fields* are

$$b(X) = C(0|X) \tag{1.2}$$

and

$$f_A(X) = \frac{\partial}{\partial Y^A} C(Y|X)\Big|_{Y=0} . \tag{1.3}$$

As a consequence of (1.1), they satisfy, respectively, the equations

$$\left(\frac{\partial^2}{\partial X^{AB} \partial X^{CD}} - \frac{\partial^2}{\partial X^{AC} \partial X^{BD}} \right) b(X) = 0 \tag{1.4}$$

and

$$\frac{\partial}{\partial X^{AB}} f_C(X) - \frac{\partial}{\partial X^{AC}} f_B(X) = 0 . \tag{1.5}$$

All the fields

$$C_{A_1 \dots A_n}(X) = \frac{\partial^n}{\partial Y^{A_1} \dots \partial Y^{A_n}} C(Y|X)\Big|_{Y=0} , \qquad n > 1 \tag{1.6}$$

are auxiliary, being expressed via higher X-derivatives of the dynamical fields by virtue of the equations (1.1). In [4] it was shown that the equations (1.4) and (1.5) along with constraints that express the auxiliary fields via X-derivatives of the dynamical fields exhaust the content of the unfolded system (1.1). That the equations (1.1) formulated in the ten dimensional space-time still describe massless fields in four dimensions was also shown in [4] using the unfolded dynamics approach.

The description of infinite towers of $4d$ massless fields in $\mathcal{M}_M$ is compact, efficient and have a number of remarkable properties. The related issues have been studied in a number of papers [10–22]. In our recent paper [22] the form of the unfolded field equations for conserved higher-spin currents was worked out. In particular, it was shown that the consistent formulation of currents requires extension of the equations (1.1) to complex coordinates

$$\mathcal{Z}^{AB} = X^{AB} + i\,\mathbf{X}^{AB} \equiv \operatorname{Re} \mathcal{Z}^{AB} + i \operatorname{Im} \mathcal{Z}^{AB}, \tag{1.7}$$

$$\mathcal{Y}^{A} = Y^{A} + i\,\mathbf{Y}^{A} \equiv \operatorname{Re} \mathcal{Y}^{A} + i \operatorname{Im} \mathcal{Y}^{A}. \tag{1.8}$$

The real part of $\mathcal{Z}^{AB}$ is identified with coordinates of the ten-dimensioned generalized space-time X^{AB} that contain in particular Minkowski coordinates. The imaginary part $\mathbf{X}^{AB} = \operatorname{Im} \mathcal{Z}^{AB}$ is required to be positive definite and was treated in [10] as a regulator that makes the Gaussian integrals well-defined (i.e., physical quantities are obtained in the limit $\mathbf{X}^{AB} \to 0$; note, that the complex coordinates Z^{AB} of [10] are related to $\mathcal{Z}^{AB}$ via $\mathcal{Z}^{AB} = i\overline{Z}^{AB}$). The space of coordinates $\mathcal{Z}^{AB}$ forms the upper Siegel half-space $\mathfrak{H}_M$ [23–25]. Evidently, $-\overline{\mathcal{Z}}^{AB} \in \mathfrak{H}_M$ provided that $\mathcal{Z}^{AB} \in \mathfrak{H}_M$, and vice versa.

In [22] it was shown that complex conjugated holomorphic and antiholomorphic in $\mathfrak{H}_M$ solutions of the complexified equations (1.1) with $\mu = \pm i\hbar$

$$\left(\frac{\partial}{\partial \mathcal{Z}^{AB}} + i\,\hbar \frac{\partial^2}{\partial \mathcal{Y}^A \partial \mathcal{Y}^B}\right) C^+(\mathcal{Y}|\mathcal{Z}) = 0\,, \tag{1.9}$$

$$\left(\frac{\partial}{\partial \overline{\mathcal{Z}}^{AB}} - i\,\hbar \frac{\partial^2}{\partial \overline{\mathcal{Y}}^A \partial \overline{\mathcal{Y}}^B}\right) C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}}) = 0 \tag{1.10}$$

$\left(\overline{C^+(\mathcal{Y}|\mathcal{Z})} = C^-(\overline{\mathcal{Y}}|\overline{\mathcal{Z}})\right)$ describe positive- and negative-frequency solutions of the massless field equations. It should be noted that $\mathfrak{H}_M$ is a $\mathrm{Sp}(2M)$ invariant homogeneous space $\mathrm{Sp}(2M|\mathbb{R})/\mathcal{U}(M)$ [24]. A closely related intriguing and, most likely, important observation is that solutions of the positive-frequency unfolded equations (1.9) with $\hbar = \dfrac{1}{4\pi}$ include Riemann theta-function [22]

$$C^+(\mathcal{Y}|\mathcal{Z}) = \sum_{n^A \in \mathbb{Z}^M} \exp i(\pi \mathcal{Z}^{AB} n_A n_B + 2\pi n_C \mathcal{Y}^C)\,, \tag{1.11}$$

which is the D-function for the subclass of solutions periodic in the auxiliary variables $\mathcal{Y}^A$. Indeed, Riemann theta-function is defined in the upper Siegel half-space $\mathfrak{H}_M$ and $\mathrm{Sp}(2M|\mathbb{Z})$ plays the fundamental role in its theory [24, 25]. This observation raises the question of the geometric interpretation of the variables $\mathcal{Y}^A$ on the same footing as $\mathcal{Z}^{AB}$. This question drives us immediately to the twistor theory [2, 26, 27] (also see [28] and references therein for its applications in $N = 4$ Yang-Mills and string theory). One of the aims of this paper is to clarify the $\mathrm{Sp}(2M)$ invariant twistor construction which in some important detail differs from the standard $\mathrm{SU}(2,2)$ (i.e., $4d$ conformal) setup [26, 27], that corresponds to the case of $M = 4$. More generally, the original motivation for this work was to elaborate a coordinate independent global description of massless fields in Lagrangian Grassmannian.

In fact it is a folklore statement that there should be a deep relation between unfolded dynamics and twistor theory, which, to the best of our knowledge, has been never clearly spelled out so far. In this paper we discuss this correspondence in some more detail.

Indeed, in the problem of interest, it is self-suggestive that the variables Y $\left(\text{or } \mathcal{Y}\right)$ are analogous to the twistor coordinates in the usual twistor theory. X^{AB} $\left(\mathcal{Z}^{AB}\right)$ are coordinates of generalized (complexified) Minkowski space $\mathcal{M}_M$ $\left(\mathfrak{H}_M\right)$. Together, (X^{AB}, Y^A) $\left((\mathcal{Z}^{AB}, \mathcal{Y}^A)\right)$ are local coordinates of the correspondence space $\mathbf{C}$. The unfolded field equation should encode the integral Penrose transform. Indeed, generic solution of (1.1) can locally be given in the form [4]

$$C(Y|X) = \exp\left(-\mu X^{AB}\frac{\partial^2}{\partial Y^A \partial Y^B}\right) C(Y|0)\,, \tag{1.12}$$

where the "initial data" $C(Y|0)$ is an arbitrary function of M variables Y^A. Then, generic solution of massless field equations (1.4) and (1.5) is determined by an arbitrary (unrestricted) function $C(Y|0)$ on the twistor space $\mathbf{T}$ with local coordinates Y according to (1.2) and (1.3). These steps should provide a $\mathrm{Sp}(2M)$ realization of a version of Penrose transform, expressed by the diagram

$$\begin{tikzcd} & \mathbf{C} \arrow[dl, "\eta"'] \arrow[dr, "\nu"] & \\ \mathbf{M} & & \mathbf{T} \end{tikzcd}\quad , \tag{1.13}$$

where $\mathbf{M} = \mathrm{Sp}(2M)/P$.

There is however important difference with the standard twistor description of massless fields in usual (compactified) Minkowski space. Indeed, the reduction of the equation (1.1) to Minkowski space is

$$\left(\frac{\partial}{\partial X^{\alpha\alpha'}} + \mu\frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\alpha'}}\right) C(y|X) = 0\,, \tag{1.14}$$

where $\alpha, \beta \ldots 1, 2$ and $\alpha', \beta' \ldots 1, 2$ are two-component indices with the convention that $A = (\alpha, \alpha')$ and $Y^A = (y^\alpha, \bar{y}^{\alpha'})$. Apart from expressing auxiliary fields (i.e., those depended both on y^α and $\overline{y}^{\alpha'}$) in terms of space-time derivatives of the (anti)holomorphic fields, the equation (1.14) imposes the massless field equations on the latter

$$\frac{\partial}{\partial y^{[\alpha}}\frac{\partial}{\partial X^{\beta]\alpha'}} C(y,0|X) = 0\,, \qquad \frac{\partial}{\partial \bar{y}^{[\alpha'}}\frac{\partial}{\partial X^{\alpha\beta']}} C(0,\bar{y}|X) = 0 \tag{1.15}$$

and

$$\Box C(0,0|X) = 0\,. \tag{1.16}$$

The equation (1.14) is transformed to the first-order equation by a half-Fourier transform with respect to $\bar{y}^{\alpha'}$ such that

$$\frac{\partial}{\partial \bar{y}^{\alpha'}} \longrightarrow i\pi_{\alpha'}\,. \tag{1.17}$$

In these terms the equation (1.14) for the Fourier transformed field $\widetilde{C}(y, \pi|X)$ takes the form

$$\left(\frac{\partial}{\partial X^{\alpha\alpha'}} + \pi_{\alpha'}\frac{\partial}{\partial y^\alpha}\right) \widetilde{C}(y, \pi|X) = 0 \tag{1.18}$$

with the appropriate choice of μ.

Since

$$C(y,\bar y|X) = \int d^2\pi\ \exp(i\pi_{\alpha'}\bar y^{\alpha'})\widetilde{C}(y,\pi|X)\,,$$

from (1.12) one has

$$\begin{aligned} C(y,\bar y|X) &= \int d^2\pi\ \exp(i\pi_{\alpha'}\bar y^{\alpha'})\exp\left(-i\mu X^{\alpha\beta'}\frac{\partial}{\partial y^\alpha}\pi_{\beta'}\right)\widetilde{C}(y,\pi|0) = \\ &= \int d^2\pi\ \exp(i\pi_{\alpha'}\bar y^{\alpha'})\widetilde{C}(y^\alpha - i\mu X^{\alpha\beta'}\pi_{\beta'},\pi|0)\,. \end{aligned} \tag{1.19}$$

At $y=\bar y=0$ this formula reproduces the nonprojective version of the Penrose formula for massless fields [26].

Since the Fourier transformed equation (1.18) is of first-order it can be interpreted as the condition

$$JC = 0 \tag{1.20}$$

for some vector field J on functions on the correspondence space. This effectively reduces functions on the correspondence space to those on the twistor space. The Fourier transform (1.17) distinguishes between primed and unprimed spinors in the twistor construction. Although somewhat ugly, this is standard in twistor theory. In the case of $\mathfrak{sp}(8)$ invariant equations this trick it is not possible. Indeed, even after a half-Fourier transform at least some of the unfolded equations in $\mathcal{M}_M$ contain second Y-derivatives, $\frac{\partial^2}{\partial y^\alpha\partial y^\beta}$ or $\frac{\partial^2}{\partial y^{\alpha'}\partial y^{\beta'}}$.

Another important difference between the Sp(8) invariant construction of infinite sets of higher-spin fields and the standard one used for particular massless fields is that in the latter case the helicity group $\mathcal{U}(1)\subset \mathrm{Sp}(8)$ distinguishes between different massless fields (i.e., spins). The helicity generator is

$$\mathcal{H} = y^\alpha\frac{\partial}{\partial y^\alpha} - \bar y^{\alpha'}\frac{\partial}{\partial \bar y^{\alpha'}}\,. \tag{1.21}$$

The centralizer of $\mathcal{H}$ in $\mathfrak{sp}(8)$ is $\mathfrak{u}(2,2)=\mathfrak{su}(2,2)\oplus\mathfrak{u}(1)$, where $\mathfrak{u}(1)$ is generated by $\mathcal{H}$. Fields associated to a given spin have definite homogeneity

$$\mathcal{H}C(Y) = \pm 2\,s\,C(Y)\,. \tag{1.22}$$

This means that the true coordinates appropriate for the description of a massless field of definite helicity are of the projective space invariant under the action of $\mathcal{H}$

$$y^\alpha \longrightarrow \exp(i\phi)y^\alpha\,,\qquad \bar y^{\alpha'}\longrightarrow \exp(-i\phi)\bar y^{\alpha'}\,, \tag{1.23}$$

equivalent to

$$y^\alpha \longrightarrow \exp(i\phi)y^\alpha\,,\qquad \pi_{\alpha'}\longrightarrow \exp(i\phi)\pi_{\alpha'}\,. \tag{1.24}$$

In the case where all fields are involved, the condition (1.22) has to be relaxed. As a result, $(y^\alpha,\pi^{\alpha'})$ become local (not homogeneous) coordinates of the appropriate twistor space, as was realized by many authors (see e.g. [3]). This has two different consequences.

The relevant symmetry group G becomes not reductive and the twistor space becomes noncompact. Simultaneously, the system becomes Sp(8) rather than SU(2, 2) symmetric.

In this paper we suggest a geometric realization of the diagram (1.13), that underlies the Sp(8) invariant systems, as well as their Sp($2M$) generalizations. The appropriate construction results from the coset space $SpH(2M)/P$. Here SpH is a semidirect product of Sp($2M$) with the Heisenberg group H_M with $2M$ noncentral elements (i.e., M pairs of oscillators). As such it is not reductive. P is some quasiparabolic subgroup of $SpH(2M)$.

To achieve a coordinate independent description we find it convenient to use the BRST language. Namely, the quotient space can be effectively described by the invariance condition

$$Q_P\, f = 0\,, \tag{1.25}$$

where Q_P is the canonical BRST operator of the group P. Indeed, for 0-forms f this condition implies that f is constant on any orbit of P, i.e., it is a function on $SpH(2M)/P$. Since Q_P is built from right vector fields of P on $SpH(2M)$, it is globally defined on $SpH(2M)$. The key observation is that, in the local coordinates X^{AB} and Y^A, the first term in eq. (1.1) can be interpreted as the remaining right vector field J_{AB} on Sp($2M$) while the second terms amounts to the bilinear combination $J_A J_B$ of the remaining vector fields on the Heisenberg group. We therefore look for a BRST operator $\mathbf{Q}$ constructed in terms of $SpH(2M)$ right Lie vector fields such that $\mathbf{Q}^2 = 0$ and the condition $\mathbf{Q}f = 0$ gives an extension of the equation (1.1) to $SpH(2M)$. The construction of this BRST operator is the main result of this paper, which provides in particular the global description of the system.

The rest of the paper is organized as follows. We start in section 2 by recalling the construction of Lagrangian Grassmannian as a quotient space. In subsections 2.1 and 2.2, we consider, respectively, the real and complex cases. In subsection 2.2, the realization of the complex Siegel space as an Sp($2M|\mathbb{R}$) orbit in the complex Lagrangian Grassmannian is discussed. In subsection 2.3, we recall the definition of real and complex Heisenberg group and introduce higher rank Heisenberg extensions of Sp($2M|\mathbb{R}$) and Sp($2M|\mathbb{C}$) as well as higher rank Fock-Siegel spaces. In section 3, we recall some elementary facts on BRST operators. In section 4, higher rank nonstandard $SpH(2M)$ invariant BRST operators $\mathbf{Q}_r$ are introduced. In section 5, right invariant vector fields on $SpH_r(2M)$ are presented. In section 6, the higher-spin unfolded equations are obtained from $\mathbf{Q}_1 f = 0$. Analogously, in section 7, the higher-spin current equations are obtained from $\mathbf{Q}_2 f = 0$.

2 Homogeneous manifolds

2.1 Real Lagrangian Grassmannian

Let us recall the quotient space construction of Lagrangian Grassmannian. The group Sp($2M|\mathbb{R}$) is constituted by real matrices

$$G = \begin{pmatrix} a^A{}_B & b^{AD} \\ c_{CB} & d_C{}^D \end{pmatrix} \tag{2.1}$$

with $M \times M$ blocks $a^A{}_B$, b^{AB}, c_{AB}, $d_A{}^B$ that satisfy the relations

$$a^A{}_C b^{DC} - a^D{}_C b^{AC} = 0, \quad a^A{}_C d_B{}^C - b^{AC} c_{BC} = \delta^A{}_B, \quad c_{BC} d_A{}^C - c_{AC} d_B{}^C = 0 \quad (2.2)$$

equivalent to the invariance condition $AJA^t = J$ for the symplectic form

$$J = \begin{pmatrix} 0 & I^A{}_B \\ -I_C{}^D & 0 \end{pmatrix}, \tag{2.3}$$

where I is the unit $M \times M$ matrix and A^t is a transposed matrix. Note, that from (2.2) it follows that

$$G^{-1} = \begin{pmatrix} a'^A{}_B & b'^{AD} \\ c'_{CB} & d'_C{}^D \end{pmatrix} = \begin{pmatrix} d_B{}^A & -b^{DA} \\ -c_{BC} & a^D{}_C \end{pmatrix}. \tag{2.4}$$

$\mathrm{Sp}(2M|\mathbb{R})$ contains the following important subgroups. The Abelian subgroup of translations T consists of the elements

$$t(X) = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \tag{2.5}$$

with various $X^{AB} = X^{BA}$. The product law in T is $t(X)t(Y) = t(X+Y)$.

Analogously, the Abelian subgroup S of special conformal transformations is constituted by the matrices (2.1) with $a = d = I$, $b = 0$. The subgroup $GL(M)$ of generalized Lorentz transformations $\mathrm{SL}(M)$ and dilatations consists of the matrices (2.1) with $b = c = 0$ and $a^B{}_C d_A{}^C = \delta^B_A$.

The parabolic subgroup $P(\mathbb{R})$ of $\mathrm{Sp}(2M|\mathbb{R})$ relevant to our consideration is the closure of S and $GL(M)$, i.e.,

$$P \ni p = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}. \tag{2.6}$$

It is a maximal parabolic subgroup. In notations of [2] it is depicted by the diagram  that results from crossing out the right node of the Dynkin diagram of $Sp(2M|\mathbb{R})$.

Lagrangian Grassmannian $\mathcal{CM}_M$ is the homogeneous space

$$\mathcal{CM}_M = \mathrm{Sp}(2M|\mathbb{R})/P(\mathbb{R}),$$

i.e., it is constituted by the elements $h \in \mathrm{Sp}(2M|\mathbb{R})$ identified modulo the right action of $P(\mathbb{R})$

$$h \sim h_1 = hp, \qquad h \in \mathrm{Sp}(2M|\mathbb{R}), \quad p \in P(\mathbb{R}).$$

Any $G \in \mathrm{Sp}(2M|\mathbb{R})$ with nondegenerate d (2.1) can be represented in the form

$$\begin{pmatrix} a^A{}_B & b^{AC} \\ c_{DB} & d_D{}^C \end{pmatrix} = \begin{pmatrix} \delta^A{}_E & X^{AF} \\ 0 & \delta_D{}^F \end{pmatrix} \begin{pmatrix} \mathcal{A}^E{}_G & 0 \\ 0 & \mathcal{D}_F{}^H \end{pmatrix} \begin{pmatrix} \delta^G{}_B & 0 \\ \mathcal{C}_{HB} & \delta_H{}^C \end{pmatrix}. \tag{2.7}$$

JHEP12(2009)021

This gives

$$\begin{pmatrix} a^A{}_B & b^{AC} \\ c_{DB} & d_D{}^C \end{pmatrix} = \begin{pmatrix} \mathcal{A}^A{}_B + X^{AF}\mathcal{D}_F{}^G\mathcal{C}_{GB} & X^{AF}\mathcal{D}_F{}^C \\ \mathcal{D}_D{}^G\mathcal{C}_{GB} & \mathcal{D}_D{}^C \end{pmatrix} ,$$

where

$$\mathcal{A}^A{}_B = (d^{-1})_B{}^A\,, \quad X^{A\,D} = b^{AC}\mathcal{A}^D{}_C, \quad \mathcal{C}_{B\,A} = c_{CB}\mathcal{A}^C{}_A \tag{2.8}$$

can be chosen as local coordinates on $\mathrm{Sp}(2M|\mathbb{R})$, where $X^{B\,A} = X^{A\,B}$ and $\mathcal{C}_{B\,A} = \mathcal{C}_{A\,B}$ by virtue of the identities

$$-\,c_{B\,A}d^{-1}{}_C{}^B + c_{B\,C}d^{-1}{}_A{}^B = 0\,, \quad -b^{AB}d^{-1}{}_B{}^C + b^{C\,B}d^{-1}{}_B{}^A = 0,$$

which follow from (2.2) and (2.4). Note that $\mathcal{D}_A{}^B = d_A{}^B$.

The big cell $\mathcal{M}_M$ of $\mathcal{CM}_M$ consists of the classes represented by elements $t(X)$ of the group of translations T. $\mathrm{Sp}(2M|\mathbb{R})$ acts in $\mathcal{M}_M$ by the Möbius transformation

$$X' = (AX+B)(CX+D)^{-1} \quad , \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \in \mathrm{Sp}(2M|\mathbb{R}). \tag{2.9}$$

By virtue of (2.7) any element (2.1) of $\mathrm{Sp}(2M|\mathbb{R})$ with $det|d^A{}_B| \neq 0$ belongs to some equivalence class associated to a point of $\mathcal{M}_M$. From (2.2), (2.6) it follows that d is non-degenerate for any $p \in P(\mathbb{R})$. As a result, the rank of the block d of a given element $g \in \mathrm{Sp}(2M|\mathbb{R})$ (2.1) is the same for all gP. Let $rank(g) = rank\,|d|\ \forall g \in \mathcal{CM}_M$. $rank(g)$ characterizes different types of equivalence classes, i.e., different subsets of the compactified space-time $\mathcal{CM}_M$. Those with $rank(g) < M$ correspond to generalized conformal infinity.

2.2 Complex Lagrangian Grassmannian and Siegel space

Complex Lagrangian Grassmannian $\mathcal{CM}_M^{\mathbb{C}} = \mathrm{Sp}(2M|\mathbb{C})/P(\mathbb{C})$ results from the analogous construction for $\mathrm{Sp}(2M|\mathbb{C})$ and its complex parabolic subgroup $P(\mathbb{C})$. Local coordinates of $\mathcal{CM}_M^{\mathbb{C}}$ will be denoted by Z. The $\mathrm{Sp}(2M|\mathbb{C})$ Möbius transformation has the same form (2.9)

$$Z' = (AZ+B)(CZ+D)^{-1}\,, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \in \mathrm{Sp}(2M|\mathbb{C}). \tag{2.10}$$

Being invariant under $\mathrm{Sp}(2M|\mathbb{R}) \subset \mathrm{Sp}(2M|\mathbb{C})$, $\mathcal{CM}_M^{\mathbb{C}}$ is not $\mathrm{Sp}(2M|\mathbb{R})$ homogeneous, containing different orbits.

The following three orbits of $\mathrm{Sp}(2M|\mathbb{R})$ are of most interest for us:

I. $\mathcal{CM}_M$.

II. The upper Siegel half-space $\mathfrak{H}_M$ [23] of matrices $\mathcal{Z}^{AB}$ with positive definite $\mathrm{Im}\,\mathcal{Z}^{AB}$.

III. The lower Siegel half-space $\mathfrak{H}_M{}^-$ of matrices $\mathcal{Z}^{AB}$ with negative definite $\mathrm{Im}\,\mathcal{Z}^{AB}$.

Note, that both $\mathfrak{H}_M$ and $\mathfrak{H}_M{}^-$ can be realized as $\mathrm{Sp}(2M|\mathbb{R})/\mathcal{U}(M)$ [23–25].

JHEP12(2009)021

2.3 Heisenberg extension

The $(2M+1)-$dimensional Heisenberg group $H_M = \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^1$ constituted by

$$\mathcal{F} = \{\mathtt{f}\,,\, u\} \qquad \mathtt{f} = y^A\,,\, w_B \qquad A,\; B = 1,\ldots,M \tag{2.11}$$

has the product law

$$\mathcal{F}_1 \circ \mathcal{F}_2 = \{\mathtt{f}_1 + \mathtt{f}_2\,,\; u_1 + u_2 - (\mathtt{f}_1\,,\, \mathtt{f}_2)\},$$

where $(\,,\,)$ is the symplectic form

$$(\mathtt{f}_1\,,\, \mathtt{f}_2) = {y_1}^A\; w_{2A} - {y_2}^A\; w_{1A}\; = -(\mathtt{f}_2\,,\, \mathtt{f}_1)\,, \qquad A = 1,\ldots,M. \tag{2.12}$$

The Heisenberg group $H_M(\mathbb{R})$ contains two quasiparabolic subgroups $H^{\pm}_M(\mathbb{R})$

$$H^- = \{\,0\,,\, w_A\;,\, u\}, \qquad H^+ = \{y^A\,,\, 0\;,\, u\}. \tag{2.13}$$

The respective quotient spaces $H_M\,/H^{\pm}$ are $R^{\pm} \sim \mathbb{R}^M$. Note that the appropriate spaces of functions on $R^{\pm}$ form Fock spaces. Complexification of this construction is straightforward, $H_M(\mathbb{C}) = \mathbb{C}^M \times \mathbb{C}^M \times \mathbb{C}^1$, etc.

$\mathrm{Sp}(2M|\mathbb{R})$ $\Big(\mathrm{Sp}(2M|\mathbb{C})\Big)$ acts canonically on H_M $\Big(H_M(\mathbb{C})\Big)$ which is the manifestation of the standard fact that $\mathrm{Sp}(2M)$ possesses the oscillator realization (see e.g. [29]). This makes it possible to introduce a semi-direct product $SpH(2M) = \mathrm{Sp}(2M) \ltimes H_M$

$$SpH(2M)\;:\;\mathcal{G} = \{G\,,\,\mathcal{F}\}\,, \qquad G \in \mathrm{Sp}(2M), \quad \mathcal{F} = \{\mathtt{f}\,,\, u\} \in H_M \tag{2.14}$$

with the product law

$$\mathcal{G}_1 \circ \mathcal{G}_2 = \{G_1 G_2\,,\;\; \mathtt{f}_1\; + \; G_1\mathtt{f}_2\;,\;\; u_1\; + \; u_2\; - \; (\mathtt{f}_1\,, G_1\mathtt{f}_2)\},$$

where $(\,,\,)$ is the symplectic form (2.12) and

$$G\mathtt{f} = \begin{pmatrix} a^A{}_B & b^{AD} \\ c_{CB} & d_C{}^D \end{pmatrix}\begin{pmatrix} y^B \\ w_D \end{pmatrix} = \begin{pmatrix} a^A{}_B y^B + b^{AD} w_D \\ c_{CB} y^B + d_C{}^D w_D \end{pmatrix},\; G \in \mathrm{Sp}(2M),\;\; \mathtt{f} = (y^A, w_B) \in H_M.$$

Analogously, over any field $\mathbb{A}$ and for any natural r we introduce a rank r Heisenberg extensions $SpH_r(2M|\mathbb{A})$ as

$$SpH_r(2M|\mathbb{A}) = \mathrm{Sp}(2M|\mathbb{A}) \ltimes \underbrace{H_M(\mathbb{A}) \times \cdots \times H_M(\mathbb{A})}_{r}. \tag{2.15}$$

Note, that $SpH_1(2M) \equiv SpH(2M)$. When it does not lead to misunderstandings, we will use shorthand notation like SpH instead of $SpH(2M|\mathbb{R})$ or $SpH(2M|\mathbb{C})$ and PH instead of $PH(2M|\mathbb{R})$ or $PH(2M|\mathbb{C})$ etc.

Consider the lower quasiparabolic subgroup $PH(2M|\mathbb{R}) = P \ltimes H^- \subset SpH(2M|\mathbb{R})$

$$PH(2M|\mathbb{R}) = \left\{ \begin{pmatrix} p^A{}_B & 0 \\ p_{CB} & p_C{}^D \end{pmatrix},\, 0\,,\, p_A\,,\, p \right\}.$$

Local coordinates on SpH/PH can be chosen as

$$X^{AB}\,, \quad Y^A = y^A - w_B X^{AB}. \tag{2.16}$$

Analogously we define SpH_r/PH_r with local coordinates X^{AB} and $Y_1{}^A,\dots,Y_r{}^A$, and their complexifications with local coordinates Z^{AB} and $\mathcal{Y}_1{}^A,\dots,\mathcal{Y}_r{}^A$.

The orbit of $\mathrm{Sp}(2M|\mathbb{R})$ in $SpH(2M|\mathbb{C})/PH(2M|\mathbb{C})$ with positive definite $\operatorname{Im} Z^{AB}$ is the upper Fock-Siegel space $\mathbb{C}^M\times\mathfrak{H}_M$ introduced in [22]. The orbit with negative definite $\operatorname{Im} Z^{AB}$ is the lower Fock-Siegel space $\mathbb{C}^M\times\mathfrak{H}_M^-$. Consider a space U of functions[1] $C(\mathcal{G})$ on SpH/PH valued in a one dimensional $PH(2M|\mathbb{R})$-module V. Following standard induced module construction, require

$$C(\mathcal{G}\circ\mathcal{P}) = \det{}^{\gamma}(p^A{}_B)\exp\left(-i\frac{1}{2}\hbar p\right)C(\mathcal{G}), \qquad \mathcal{G}\in SpH/PH, \quad \mathcal{P}\in PH, \tag{2.17}$$

where the generalized conformal weight γ and Plank constant $\hbar$ are arbitrary parameters that characterize V. The space U forms a left $SpH(2M|\mathbb{R})$-module with the transformation law

$$C(\mathcal{G})\longrightarrow C_G(\mathcal{G}) = C(G\circ\mathcal{G})\,, \qquad G\in SpH(2M|\mathbb{R}).$$

In local coordinates X, Y (2.16) for G of the form (2.1) one obtains,

$$C_G(Y|X) = \det{}^{-\gamma}|cX+d|\exp\left(-\frac{1}{2}i\hbar Y'^{\,A}Y^B c_{AB}\right)C(Y'|X')\,, \tag{2.18}$$

where

$$X' = (aX+b)(cX+d)^{-1}\,, \qquad Y' = (cX+d)^{-1}Y\,.$$

For $\gamma=\dfrac{1}{2}$, (2.18) is the $\mathrm{Sp}(2M)$ transformation law for solutions of the equations (1.1) with $\mu=\frac{i}{2\hbar}$, infinitesimal form of which was obtained in [10]. This suggests that it should be possible to impose the equation (1.1) in a manifestly $\mathrm{Sp}(2M)-$invariant and coordinate independent way. This will be achieved in section 4 using BRST technics.

Analogously, the space of complex functions $\mathbf{f}(\mathcal{G})$ on the upper Fock-Siegel space $\mathbb{C}^M\times\mathfrak{H}_M\subset SpH(2M|\mathbb{C})/PH(2M|\mathbb{C})$ valued in a one dimensional $PH(2M|\mathbb{C})$ module subjected to (2.17) forms a $SpH(2M)$-module under the group action (2.18). For $G\in\mathrm{Sp}(2M|\mathbb{R})$ of the form (2.1) one obtains in local coordinates $\mathcal{Z}$, $\mathcal{Y}$

$$C_G(\mathcal{Y}|\mathcal{Z}) = C(\mathcal{Y}'|\mathcal{Z}')\det{}^{-\gamma}|c\mathcal{Z}+d|\exp\left(-i\frac{1}{2}\hbar\mathcal{Y}'^{\,A}\mathcal{Y}^B c_{AB}\right)\,, \tag{2.19}$$

where

$$\mathcal{Z}' = (a\mathcal{Z}+b)(c\mathcal{Z}+d)^{-1}\,, \qquad \mathcal{Y}' = (c\mathcal{Z}+d)^{-1}\mathcal{Y}\,. \tag{2.20}$$

This formula with $\gamma=\frac{1}{2}$ is in agrement with the fact that Riemann theta functions solve the complexified unfolded higher-spin field equations (1.1) with $\mu=\dfrac{i}{4\pi}$.

[1] Using this abused terminology to simplify the presentation, we assume that functions should be substituted by appropriate sections of fiber bundles whenever necessary.

3 Canonical BRST operator

Any Lie group G possesses two mutually commuting sets of left and right Lie vector fields J^l_B and J^r_A $(B\,,\; A=1,2,\ldots,\dim G)$, each forming Lie algebra $\mathfrak{g}$ of G

$$[J^r_A\,,J^r_B]=f_{AB}{}^E J^r_E\,,\qquad [J^l_A\,,J^l_B]=f_{AB}{}^E J^l_E\,,\qquad [J^r_A\,,J^l_B]=0\,.$$

Let I^r_A be a subset of J^r_A that corresponds to some subgroup $B\subset G$. The space of functions on G/B identifies with the space of functions on G that satisfy

$$I^r_A F(G)=0\,. \tag{3.1}$$

The algebra Lie $\mathfrak{g}$ acts on solutions of (3.1) by the left Lie vector fields J^l_D.

An important extension of (3.1) to the induced modules construction is

$$(I^r_A+T_A)F(G)=0\,, \tag{3.2}$$

where $F(G)$ is valued in some B-module V and T_A provide a representation of the Lie algebra $\mathfrak{b}$ of B on V,

$$[T_A\,,T_B]=f_{AB}{}^C T_C\,,\qquad [J^r_B\,,T_A]=0\,,\qquad [J^l_B\,,T_A]=0\,. \tag{3.3}$$

In what follows we will be interested in the particular case of this construction with a one dimensional B-module V. In this case, $F(G)$ is still valued in $\mathbb{R}$ or $\mathbb{C}$ and T_A is given by some constants associated to central elements of the grade zero part of $\mathfrak{b}$.

A useful way to impose the condition (3.2) is by introducing a canonical BRST (Chevalley-Eilenberg) operator

$$Q=c^A(I^r_A+T_A)-\frac{1}{2}c^Ac^Bb_Cf_{AB}{}^C\,,\qquad Q^2=0\,, \tag{3.4}$$

where the ghosts c^A and b_A obey the Clifford anticommutation relations

$$\{c^A\,,b_B\}=\delta^A_B\,,\qquad \{c^A\,,c^B\}=0\,,\qquad \{b_A\,,b_B\}=0\,. \tag{3.5}$$

The equation (3.2) results from the restriction to the sector of c-independent $F(G)$ of the condition

$$QF(G,c)=0\,, \tag{3.6}$$

where b_A is realized as

$$b_A=\frac{\partial}{\partial c^A}\,. \tag{3.7}$$

The BRST construction provides the extension of the equation (3.2) to the case of $F(G,c)$ that belongs to the Clifford-Fock module generated from the vacuum that satisfies $b_A|0\rangle=0$. This extension is expected to have useful applications in the context of the so-called σ_- cohomology in the unfolded dynamics approach [30] (see also [4, 15] and [31] for a review) as well as for the analysis of conserved charges by BRST methods.

The standard twistor theory diagram relates three objects: the correspondence space **C**, space-time **M** and twistor space **T**. In the standard twistor theory all of them are

JHEP12(2009)021

homogeneous spaces. In the model of interest this is not quite the case. Although the spaces $\mathbf{C}$ and $\mathbf{M}$ indeed allow the quotient space realization with

$$\mathbf{C} = SpH(2M|\mathbb{C})/PH(\mathbb{C})\,, \quad \mathbf{M} = \mathcal{CM}_M^{\mathbb{C}} = SpH(2M|\mathbb{C})/\big(P(\mathbb{C}) \otimes H_M(\mathbb{C})\big)\,, \tag{3.8}$$

this is not so for the twistor space $\mathbf{T}$ that should have Y^A as local coordinates. Indeed, it is not hard to see that no appropriate $\mathrm{Sp}(2M)$ homogeneous space with local coordinates Y^A exists. The point is that the realization of $\mathrm{Sp}(2M)$ on functions of the coordinates Y^A, that can be read of the equation (1.1) is given by a second-order differential operator that cannot result from a first-order vector field. It turns out however that appropriate spaces of functions on all three spaces $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{T}$ can be described by the BRST conditions $Qf = 0$. In the cases of $\mathbf{C}$ and $\mathbf{M}$ the Q_{PH} and $Q_{P\otimes H}$ are canonical for the groups PH and $P \otimes H_M$, respectively.

Namely, according to (3.4),

$$\begin{aligned} Q_{PH} = {} & c^A{}_B(J^B{}_A + \gamma\delta^B{}_A) + c_{AB}J^{AB} + c(J-\nu) + c_A J^A \\ & + c^A{}_B c^B{}_C b^C_A - 2c^A{}_B c_{AC} b^{BC} - c_A{}^B c_B b^A\,, \end{aligned} \tag{3.9}$$

where

$$J^A{}_B\,, \qquad J^{AB}\,, \qquad J_{AB}\,, \qquad J^A\,, \qquad J_A\,, \qquad J \tag{3.10}$$

are the right vector fields of SpH, while constants ν and γ characterize the induced PH-module in question. The operator $Q_{P\otimes H}$ is

$$\begin{aligned} Q_{P\otimes H} = {} & c^A{}_B(J^B{}_A + \gamma\delta^B{}_A) + c_{AB}J^{AB} + cJ + c_A J^A + c^A J_A \\ & + c^A{}_B c^B{}_C b^C_A - 2c^A{}_B c_{AC} b^{BC} - c_A{}^B c_B b^A + c^A c_A b + c_{AB} c^A b^B + c_A{}^B c^A b_B. \end{aligned} \tag{3.11}$$

The both are nilpotent,

$$Q^2_{P\otimes H} = Q^2_{PH} = 0\,,$$

as a consequence of the (anti)commutation relations

$$\begin{aligned} [J^A{}_B\,, J^C{}_E] &= \delta^C_B J^A{}_E - \delta^A_E J^C{}_B\,, \\ [J^A{}_B\,, J^{CE}] &= \delta^C_B J^{AE} + \delta^E_B J^{AC}\,, \\ [J^A{}_B\,, J_{CE}] &= -\delta^A_C J_{BE} - \delta^A_E J_{BC}\,, \\ [J_{AB}\,, J^{CE}] &= \delta^C_A J^E{}_B + \delta^C_B J^E{}_A + \delta^E_A J^C{}_B + \delta^E_B J^C{}_A\,, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &[J^A{}_B\,, J^C] = \delta^C_B J^A\,, && [J^A{}_B\,, J_C] = -\delta^A_C J_B\,, \\ &[J_{AB}\,, J^C] = \delta^C_B J_A + \delta^C_A J_B\,, && [J^{AB}\,, J_C] = -\delta^B_C J^A - \delta^B_C J^B\,, \\ &[J_A\,, J^B] = \delta^B_A J \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\{c^{AB}\,, b_{CE}\} = \frac{1}{2}(\delta^A_E\delta^B_C + \delta^B_E\delta^A_C)\,, \ \ \{c_{AB}\,, b^{CE}\} = \frac{1}{2}(\delta^E_A\delta^C_B + \delta^E_B\delta^C_A)\,, \\ &\{c^A{}_B\,, b^C{}_E\} = \delta^A_E\delta^C_B\,, \qquad\qquad \{c^A\,, b_C\} = \delta^A_B, \quad \{c_A\,, b^C\} = \delta^B_A, \quad \{c\,, b\} = 1. \end{aligned}$$

(Other (anti)commutation relations are zero.)

4 Nonstandard BRST operator

The main observation of this paper is that there exists a nonstandard BRST operator $\mathbf{Q}$ that is built from the right Lie vector fields and supplements the quotient conditions (3.6) with the equations (1.1). Since left Lie vector fields commute to the right ones, the resulting equations are manifestly invariant under the left action of $\mathfrak{g}$ at any point of G. Hence this construction is G invariant and coordinate independent.

$\mathbf{Q}$ has the form

$$\mathbf{Q} = Q_{PH} + \triangle\mathbf{Q}\,. \tag{4.1}$$

Here Q_{PH} is the standard BRST operator (3.9) of the parabolic subalgebra which reduces SpH to the correspondence space. The additional part

$$\begin{aligned}\triangle\mathbf{Q} = {}& c^{AB}(J_{AB}-\nu^{-1}J_AJ_B)+2c^A{}_Bc^{BC}b_{AC}-4c^{AB}c_{BC}b^C{}_A-2\nu^{-1}c^{AB}c_{AB}b+\\ &+2\nu^{-1}c^{AB}c_BbJ_A-4\nu^{-1}c^{AC}c_{CB}c_Abb^B+4\nu^{-1}c^{BC}c_{BA}b^AJ_C-4\nu^{-1}c^{BC}c_{BA}c_{CE}b^Eb^A.\end{aligned} \tag{4.2}$$

takes care of the coordinates X^{AB}, ensuring that the space $\mathbf{T}$ has local coordinates Y^A.

The main result of this paper is that $\mathbf{Q}^2 = 0$ provided that the generalized conformal weight is

$$\gamma = \frac{1}{2}\,. \tag{4.3}$$

To explain our construction, let us first consider the compatibility of the conditions $P_{\mathcal{A}}f = 0$ for the set of operators

$$P_{\mathcal{A}}: P^A{}_B = J^A{}_B + \gamma\delta^A{}_B\,,\quad P^{AB} = J^{AB},\quad P^A = J^A,\quad P = J-\nu,\quad P_{AB} = J_{AB}+\kappa J_AJ_B\,, \tag{4.4}$$

where γ, ν and κ are free parameters to be determined. Namely, we demand that the operators $P_{\mathcal{A}}$ (4.4) form a "closed algebra"

$$[P_{\mathcal{A}}, P_{\mathcal{B}}] = \phi^{\mathcal{C}}_{\mathcal{A}\mathcal{B}}P_{\mathcal{C}} \tag{4.5}$$

with some "structure functions" $\phi^{\mathcal{C}}_{\mathcal{A}\mathcal{B}}$ that may depend on the vector fields J.

Using (3.12) and (3.13) it is easy to see that the only commutation relations that do not obey (4.5) for generic values of the parameters γ, ν and κ are

$$\begin{aligned}[P_{AB},\, P^C\,] &= \kappa(\delta^C_AJ_B+\delta^C_BJ_A)(\kappa^{-1}+\nu+P)\,,\\ [P_{AB}, P^{DF}] &= \delta^D_AP_B{}^F+\delta^D_BP_A{}^F+\delta^F_AP_B{}^D+\delta^F_BP_A{}^D-\kappa(\delta^D_A\delta_B{}^F+\delta^D_B\delta_A{}^F)P\\ &\quad+\kappa(\delta^D_AJ_BP^F+\delta^D_BJ_AP^F+\delta^F_AJ_BP^D+\delta^F_BJ_AP^D)\\ &\quad-(2\gamma+\kappa\nu)(\delta^D_A\delta_B{}^F+\delta^D_B\delta_A{}^F)\,.\end{aligned}$$

Hence, $P_{\mathcal{A}}$ obey (4.5) at the condition (4.3) along with

$$\kappa = -\nu^{-1}\,. \tag{4.6}$$

In this case, the nonzero commutation relations between $P_{\mathcal{A}}$ take the form

$$\begin{aligned}
[P_{AB}, P^{DF}] =&\ \delta^D_A P_B{}^F + \delta^D_B P_A{}^F + \delta^F_A P_B{}^D + \delta^F_B P_A{}^D + \nu^{-1}(\delta^D_A \delta_B{}^F + \delta^D_B \delta_A{}^F)\, P \\
&-\nu^{-1}(\delta^D_A J_B + \delta^D_B J_A) P^F - \nu^{-1}(\delta^F_A J_B + \delta^F_B J_A) P^D\,, \\
[P^A{}_B\,, P^C{}_E] =&\ \delta^C_B P^A{}_E - \delta^A_E P^C{}_B\,, \\
[P^A{}_B\,, P^{CE}] =&\ \delta^C_B P^{AE} + \delta^E_B P^{AC}\,, \\
[P_{AB}, P^D{}_C] =&\ \delta^D_A P_{BC} + \delta^D_B P_{AC}\,, \\
[P_{AB},\, P^C\,] =&\ -\nu^{-1}(\delta^C_A J_B + \delta^C_B J_A) P, \\
[P^A{}_B, P^D] =&\ \delta^D_B P^A\,.
\end{aligned} \tag{4.7}$$

Since some of the right hand sides of the relations (4.7) contain products of J and P, that do not commute according to (4.4), (3.12) and (3.13), the relations (4.7) form a nonlinear algebra. Naively, one might think that ν^{-1} is a deformation parameter, which describes the nonlinear algebra (4.7) as a deformation of a Lie algebra at $\nu^{-1} = 0$. This is not true, however, because of the relation

$$[J_A,\, P^B] = \delta^B_A (P + \nu)\,,$$

that has to be used in the consistency check. The terms, that differ the relations (4.7) from a Lie algebra, include all ν-dependent terms along with the ν-independent right-hand side of the last of the relations (4.7). On the other hand, at $\nu^{-1} = 0$, the operators $P^A{}_B$, P^{AB}, P_{AB} and P form a Lie algebra $sp(2M) \oplus u(1)$. This implies that the BRST operator

$$\begin{aligned}
Q_{sp} =&\ c^A{}_B P^B{}_A + c^{AB} P_{AB} + c_{AB} P^{AB} + cP + \\
&+c^A{}_B c^B{}_C b^C_A - 2c^A{}_B c_{AC} b^{BC} + 2c^A{}_B c^{BC} b_{AC} - 4c^{AB} c_{BC} b^C{}_A
\end{aligned} \tag{4.8}$$

squares to zero at $\nu^{-1} = 0$, which observation simplifies the computation of $\mathbf{Q}^2$ sketched below.

That $P_{\mathcal{A}}$ satisfy (4.5) allows us to look for a nilpotent operator of the form

$$\mathbf{Q} = c^{\mathcal{A}} P_{\mathcal{A}} - \frac{1}{2} \sum_{n>0} \phi^{\mathcal{B}_1 \ldots \mathcal{B}_n}_{\mathcal{A}_1 \ldots \mathcal{A}_n \mathcal{A}_{n+1}}(J)\, c^{\mathcal{A}_1} \ldots c^{\mathcal{A}_n} c^{\mathcal{A}_{n+1}}\, b_{\mathcal{B}_1} \ldots b_{\mathcal{B}_n}\,, \tag{4.9}$$

where $\phi^{\mathcal{C}}_{\mathcal{AB}}$ are the "structure functions" of (4.5). The higher structure functions that enter the terms of higher orders in ghost fields can appear in the case of field-dependent $\phi^{\mathcal{C}}_{\mathcal{AB}}$ as was first found in the Hamiltonian analysis of supergravity [32]. A general analysis for classical Hamiltonian systems was given in [33]. The extension to the quantum case of associative algebras, which is of interest for us here, was considered in [34].

The Jacobi identity expresses the identity $[Q_0\,, Q_0^2] = 0$ for $Q_0 = c^{\mathcal{A}} P_{\mathcal{A}}$. In the case of a field-dependent structure function $\phi^{\mathcal{C}}_{\mathcal{AB}}$ it has the form

$$\Big(c^{\mathcal{A}} c^{\mathcal{B}} c^{\mathcal{C}} [P_{\mathcal{C}}\,, \phi^{\mathcal{D}}_{\mathcal{AB}}] + c^{\mathcal{A}} c^{\mathcal{B}} c^{\mathcal{C}} \phi^{\mathcal{E}}_{\mathcal{AB}}\, \phi^{\mathcal{D}}_{\mathcal{CE}} \Big) P_{\mathcal{D}} = 0\,. \tag{4.10}$$

Differently from the case of constant $\phi^{\mathcal{C}}_{\mathcal{AB}}$, the condition (4.10) does not imply that the expression in brackets is zero. Instead, it imposes a weaker condition

$$c^{\mathcal{A}} c^{\mathcal{B}} c^{\mathcal{C}} \Big([P_{\mathcal{C}}\,, \phi^{\mathcal{D}}_{\mathcal{AB}}] + \phi^{\mathcal{E}}_{\mathcal{AB}}\, \phi^{\mathcal{D}}_{\mathcal{CE}} + 2\phi^{\mathcal{DE}}_{\mathcal{ABC}} P_{\mathcal{E}} + \phi^{\mathcal{FE}}_{\mathcal{ABC}} \phi^{\mathcal{D}}_{\mathcal{FE}} \Big) = 0\,, \tag{4.11}$$

where $\phi^{\mathcal{DE}}_{\mathcal{ABC}} = -\phi^{\mathcal{ED}}_{\mathcal{ABC}}$ are some new structure coefficients that, in turn, should be determined from (4.11). Indeed, one can see that (4.10) follows from (4.11). Provided that (4.10) is true, the operator

$$\mathbf{Q} = Q_0 + Q_1 + Q_2\,,$$

where

$$Q_0 = c^{\mathcal{A}} P_{\mathcal{A}}\,, \qquad Q_1 = -\frac{1}{2} c^{\mathcal{A}} c^{\mathcal{B}} \phi^{\mathcal{C}}_{\mathcal{AB}} b_{\mathcal{C}}\,, \qquad Q_2 = -\frac{1}{2} c^{\mathcal{A}} c^{\mathcal{B}} c^{\mathcal{C}} \phi^{\mathcal{DF}}_{\mathcal{ABC}} b_{\mathcal{D}} b_{\mathcal{F}}\,,$$

is nilpotent up to the terms $\mathbf{Q}^2 = O(c^4 b^2)$. Generally, these terms are compensated by the terms with higher structure coefficients in (4.9). Fortunately, in the case of interest, the terms of higher orders in c and b in $\mathbf{Q}$ (4.9) are not needed because the $c^4 b^2$-type terms cancel out.

To check nilpotency of $\mathbf{Q}$, one starts with $Q_0 + Q_1$ that accounts for all terms linear and quadratic in $c^{\mathcal{A}}$

$$Q_0 + Q_1 = Q_{sp} + c_A P^A - c_A{}^B c_B b^A - 2\nu^{-1} c^{AB} c_{AB} b + 4\nu^{-1} c^{AC} c_{AB} b^B J_C + 2\nu^{-1} c^{AB} c_B b J_A,$$

where Q_{sp} is given in (4.8). Taking into account that Q^2_{sp} only contains terms that are zero at $\nu^{-1} = 0$, it is not hard to obtain

$$(Q_0 + Q_1)^2 = 4\nu^{-1} c_{AB} c^{AC} (-c_C b P^B - 2 c_{CD} b^D P^B + c_C b^B P)\,.$$

These terms are cancelled by

$$Q_2 = -4\nu^{-1} (c^{AB} c_{BC} c_A b b^C + c^{AC} c_{AE} c_{CB} b^E b^B)\,.$$

Namely,

$$\{Q_0 + Q_1,\, Q_2\} = -(Q_0 + Q_1)^2\,,$$

and

$$(Q_2)^2 = (4\nu)^{-2} c_{DE} c^{DF} c_{FA} c^{AB} c_{BC} b^E b^C b = 0$$

because the tensor $F_{EC} = c_{DE} c^{DF} c_{FA} c^{AB} c_{BC}$ is symmetric in the indices E and C.

Let us comment on the relation of our construction with the BRST operators that follow from the constraint algebra of twistor particle models considered for example in [3, 4, 35–37]. Roughly speaking, the latter correspond to the first term of the operator (4.2) i.e., to those terms that only contain the ghost c^{AB}. The extension to its Heisenberg algebra counterpart was discussed in [38]. Reductions of $\mathbf{Q}$ of this type trivially square to zero. Usually such operators are given in particular coordinates like X^{AB} and Y^A. The operator $\mathbf{Q}$ constructed in this paper extends this construction in two important respects. Firstly, it extends the construction to full $\mathrm{Sp}(2M)$. This in particular determines the generalized $\mathrm{Sp}(2M)$ conformal weight γ of the massless fields in $\mathcal{M}_M$. Secondly, it is formulated in a coordinate-independent way and is globally defined thus giving rise to the coordinate independent version of $\mathrm{Sp}(2M)$ invariant unfolded equations (1.1). Indeed, to work in any

JHEP12(2009)021

coordinate system it suffices to substitute the corresponding expressions for the right Lie vector fields into $\mathbf{Q}$.

Otherwise the proposed BRST operator differs from most versions of the BRST constructions used in higher-spin theory (see e.g. [39–45] and references therein). In our approach, we do not introduce *ad hoc* any auxiliary space and constraints to guess a BRST operator that gives rise to appropriate dynamics, working directly in terms of the symmetry group and its Lie vector fields, that makes the setting manifestly symmetric, coordinate independent and globally defined. Let us stress that it is impossible to introduce a deformation parameter into $\mathbf{Q}$ to treat it as a deformation of a BRST operator associated to some Lie algebra. (Note that there is some similarity with the BRST description of higher spins in AdS background [40], where however, the BRST operator is a deformation of the standard one in Minkowski space.)

The generalization of $\mathbf{Q}$ (4.1) to the case of the rank r quasiparabolic subgroup $P \otimes \underbrace{{H_M}^- \times \cdots \times {H_M}^-}_{r}$ is straightforward

$$\begin{aligned}\mathbf{Q}_r = & \; c^A{}_B P^B{}_A + c_{AD}P^{AD} + c^{AB}P_{AB} + \sum_{j=1}^{r}\left(c_j P_j + c_{jA}{P_j}^A\right)\\ & + c^A{}_B c^B{}_C b^C_A - 2c^A{}_B c_{AC} b^{BC} + 2c^A{}_B c^{BC} b_{AC} - 4c^{AB} c_{BC} b^C{}_A\\ & - \sum_{j=1}^{r} c_A{}^B c_{jB} {b_j}^A + \sum_{j=1}^{r}\nu_j^{-1}\Big(2c^{AB}c_{AB}b_j + 2c^{AB}c_{jB}b_j J_{jA} + 4c^{AB}c_{AC}{b_j}^C J_{jB}\\ & \qquad\qquad\qquad\qquad\qquad - 4c^{AB}c_{BC}c_{jA}b_j {b_j}^C - 4c^{AB}c_{AC}c_{BE}{b_j}^C {b_j}^E\Big)\,,\end{aligned}$$

where

$$\begin{aligned}P_{\mathcal{A}}: \quad & P^A{}_B = J^A{}_B + \gamma_r \delta^A{}_B, \quad P^{DB} = J^{DB}, \quad P^A_i = J^A_i, \quad P_i = J_i - \nu_i\\ & P_{AB} = J_{AB} + \sum_{i=1}^{r}\kappa_i J_{iA} J_{iB}\end{aligned} \tag{4.12}$$

and

$$\kappa_i = -{\nu_i}^{-1}\,, \qquad \gamma_r = \frac{r}{2}\,. \tag{4.13}$$

Nonzero anticommutation relations of the Clifford ghosts are

$$\{{c_j}^A\,, b_{kB}\} = \delta^A_B\delta_{jk}\,, \quad \{c_{jA}\,, {b_k}^B\} = \delta^B_A\delta_{jk}\,, \quad \{c_j\,, b_k\} = \delta_{jk}\,.$$

One can make sure that $\left(\mathbf{Q}_r\right)^2 = 0$. Indeed, using that $[J_{iB}, {J_j}^D] = 0$ at $i \neq j$, it is easy to see that operators (4.12) form a "closed algebra" with nonzero commutation relations

$$
\begin{aligned}
[P^A{}_B\,, P^C{}_E] &= \delta^C_B P^A{}_E - \delta^A_E P^C{}_B\,,\\
[P^A{}_B\,, P^{CE}] &= \delta^C_B P^{AE} + \delta^E_B P^{AC}\,,\\
[P_{AB}, P^{DF}] &= \delta^D_A P_B{}^F + \delta^D_B P_A{}^F + \delta^F_A P_B{}^D + \delta^F_B P_A{}^D + \sum_j \nu_j^{-1} P_j (\delta^D_A \delta_B{}^F + \delta^D_B \delta_A{}^F)\\
&\quad - \sum_j \nu_j^{-1} (\delta^D_A J_{jB} J_j{}^F + \delta^D_B J_{jA} J_j{}^F + \delta^F_A J_{jB} J_j{}^D + \delta^F_B J_{jA} J_j{}^D)\,,\\
[P_{AB}, P^D{}_C] &= \delta^D_A P_{BC} + \delta^D_A P_{AC}\\
[P_{AB}, P^C_j] &= (\delta^C_A J_{jB} + \delta^C_B J_{jA}) P_j\,,\\
[P^A_B, P^D_j] &= \delta^D_B P^A_j.
\end{aligned}
$$

provided that (4.13) is true. The rest of the analysis is identical to the case of rank one.

5 Vector fields on $SpH(2M)$

5.1 Rank 1

To make the equations explicit in one or another coordinate system, a concrete realization of the right Lie vector fields is needed. Straightforward computation in the local coordinates

$$
\mathcal{A}_B{}^B\,, \quad X^{AB}\,, \quad \mathcal{C}_{AB}\,, \quad y^A\,, \quad w_A\,, \quad u \tag{5.1}
$$

of $SpH(2M|\mathbb{R})$ gives

$$
\begin{aligned}
J_{AB} &= -2\mathcal{A}^E{}_A \mathcal{A}^D{}_B \frac{\partial}{\partial X^{DE}} + 2\mathcal{A}^E{}_{(B}\mathcal{C}_{A)D}\frac{\partial}{\partial \mathcal{A}^E{}_D} + 2\mathcal{C}_{AD}\mathcal{C}_{BE}\frac{\partial}{\partial \mathcal{C}_{DE}},\\
J^{AB} &= 2\frac{\partial}{\partial \mathcal{C}_{AB}},\\
J_A{}^B &= -2\mathcal{C}_{AC}\frac{\partial}{\partial \mathcal{C}_{BC}} - \mathcal{A}^C{}_A \frac{\partial}{\partial \mathcal{A}^C{}_B},
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
J^C &= \mathcal{D}_B{}^C \left(X^{BA}\frac{\partial}{\partial y^A} + \frac{\partial}{\partial w_B} + \left(-y^B + w_A X^{BA}\right)\frac{\partial}{\partial u}\right),\\
J_A &= -\mathcal{A}^B{}_A\left(\frac{\partial}{\partial y^B} + w_B \frac{\partial}{\partial u}\right) - \mathcal{C}_{AB} J^B,\\
J &= 2\frac{\partial}{\partial u}\,.
\end{aligned}
\tag{5.3}
$$

5.2 Rank r

$SpH_r(2M)$ right vector fields consist of the Sp$(2M)$ right vector fields (5.2) and r mutually commutative copies of the vector fields (5.3)

$$
\begin{aligned}
J_j{}^C &= \mathcal{D}_D{}^C \left(X^{DA}\frac{\partial}{\partial y_j{}^A} + \frac{\partial}{\partial w_{jD}} + \left(-y_j{}^D + w_{jA} X^{DA}\right)\frac{\partial}{\partial u_j}\right),\\
J_{jA} &= -\mathcal{A}^D{}_A\left(\frac{\partial}{\partial y_j{}^D} + w_{jD}\frac{\partial}{\partial u_j}\right) - \mathcal{C}_{AF} J_j{}^F,\\
J_j &= 2\frac{\partial}{\partial u_j},
\end{aligned}
\tag{5.4}
$$

where $j = 1, \dots, r$.

Let us consider more closely the case of rank two, that will be used in section 7 to construct Sp(2M) invariant *current equations*. Introducing

$$J_\pm = \frac{1}{2}(J_1 \pm J_2), \quad y_\pm = y_1 \pm y_2, \quad w_\pm = (w_1 \pm w_2), \quad u_\pm = (u_1 \pm u_2),$$

we obtain from (5.4)

$$\begin{aligned}
J_-{}^C &= \mathcal{D}_D{}^C \left(\frac{\partial}{\partial w_{-D}} + X^{DA} \frac{\partial}{\partial y_-{}^A} - \frac{1}{2} Y_+{}^D \frac{\partial}{\partial u_-} - \frac{1}{2} Y_-{}^D \frac{\partial}{\partial u_+} \right), \qquad (5.5)\\
J_{-A} &= -\mathcal{A}^D{}_A \left(\frac{\partial}{\partial y_-{}^D} + \frac{1}{2} w_{-D} \frac{\partial}{\partial u_+} + \frac{1}{2} w_{+D} \frac{\partial}{\partial u_-} \right) - \mathcal{C}_{AF} J_-{}^F,\\
J_- &= 2 \frac{\partial}{\partial u_-},\\
J_+{}^C &= \mathcal{D}_D{}^C \left(\frac{\partial}{\partial w_{+D}} + X^{DA} \frac{\partial}{\partial y_+{}^A} - \frac{1}{2} Y_+{}^D \frac{\partial}{\partial u_+} - \frac{1}{2} Y_-{}^D \frac{\partial}{\partial u_-} \right), \qquad (5.6)\\
J_{+A} &= -\mathcal{A}^D{}_A \left(\frac{\partial}{\partial y_+{}^D} + \frac{1}{2} w_{+D} \frac{\partial}{\partial u_+} + \frac{1}{2} w_{-D} \frac{\partial}{\partial u_-} \right) - \mathcal{C}_{AD} J_+{}^D,\\
J_+ &= 2 \frac{\partial}{\partial u_+},
\end{aligned}$$

where $Y_\pm{}^D = y_\pm{}^D - w_{\pm A} X^{DA}$.

6 BRST operator and unfolded equations

Let f be a function on SpH/PH, independent of the Clifford elements c and b. Let $\mathbf{Q}$ be the BRST-operator (4.1). Then the condition

$$\mathbf{Q} f = 0 \qquad (6.1)$$

implies

$$(J^B{}_A + \frac{1}{2}\delta^B{}_A) f = 0, \qquad (6.2)$$

$$J^{FD} f = 0, \qquad (6.3)$$

$$(J - \nu) f = 0, \qquad (6.4)$$

$$J^R f = 0, \qquad (6.5)$$

$$(J_{AB} - \nu^{-1} J_A J_B) f = 0. \qquad (6.6)$$

Substituting (5.2) and (5.3) into (6.2)–(6.5) we obtain

$$\left(-\mathcal{A}^C{}_B \frac{\partial}{\partial \mathcal{A}^C{}_D} + \frac{1}{2} \delta^D_B \right) f = 0\,, \qquad (6.7)$$

$$\frac{\partial}{\partial \mathcal{C}_{AB}} f = 0\,, \qquad (6.8)$$

$$\left(-2 \frac{\partial}{\partial u} + \nu \right) f = 0\,, \qquad (6.9)$$

$$\left(\frac{\partial}{\partial w_A} + X^{BA} \frac{\partial}{\partial y^B} + \left(-y^A + w_B X^{BA}\right) \frac{\partial}{\partial u} \right) f = 0\,. \qquad (6.10)$$

Note, that eqs. (6.7)–(6.10), that correspond to the subgroup $PH \subset SpH$, are first order differential equations with respect to $\mathcal{A}$, $\mathcal{C}$, u and w, respectively, thus, reconstructing the dependence on the coordinates of PH. Hence we can set $w = 0$, $\mathcal{C} = 0$ in (6.6).

Substituting (5.2), (5.3), (6.8) and (6.9) into (6.6) and taking into account (6.5) , we obtain at $w = 0$, $\mathcal{C} = 0$ in local coordinates X, Y (2.16)

$$\Big(\frac{\partial}{\partial X^{AB}} + \frac{1}{2}\nu^{-1}\frac{\partial}{\partial Y^A}\frac{\partial}{\partial Y^B}\Big) f = 0\,, \tag{6.11}$$

which is eq. (1.1) at $\mu = \frac{1}{2}\nu^{-1}$.

Since by virtue of (3.13) and (6.4) $[J_A\,, J^B] = \nu\delta_A^B$, we set

$$\nu = -i\hbar. \tag{6.12}$$

(Note that the normalization of (1.9) results from (6.11) via the rescaling $Y^A \to \frac{1}{\sqrt{2}\,\hbar}Y^A$.)

Since eq. (6.11) is of first order in X^{AB}, it reconstructs the dependence on X, i.e., solutions of (6.1) are parametrized by functions $f(Y)$ on the twistor space.

General solution of (6.1) is

$$f = \det(\mathcal{A})^{\frac{1}{2}} \exp\left(\frac{1}{2}\nu\left(u + w_B Y^B\right)\right) f_0(Y|X), \tag{6.13}$$

where $f_0(Y|X)$ is any solution of (6.11) .

Proceeding analogously with complex vector fields on $SpH(2M|\mathbb{C})$, we obtain complex unfolded equations

$$\left(\frac{\partial}{\partial Z^{AB}} + \frac{1}{2}\nu^{-1}\frac{\partial}{\partial \mathcal{Y}^A}\frac{\partial}{\partial \mathcal{Y}^B}\right) f = 0\,. \tag{6.14}$$

As mentioned in section 2, a natural complexification of generalized space-time $\mathcal{M}_M$ is the upper Fock-Siegel space $\mathbb{C}^M \times \mathfrak{H}_M \subset SpH(2M|\mathbb{C})/PH(2M|\mathbb{C})$. For $Z = \mathcal{Z} \in \mathfrak{H}_M$ eqs. (6.14) coincide with the field equations for massless fields in the Fock-Siegel space obtained in [22] up to a coefficient in front of the second term.

7 Rank 2 BRST operator and current equations

$\mathfrak{sp}(2M|\mathbb{R})$ invariant current equations introduced in [22] to formulate HS charge conservation in $\mathcal{M}_M$ have the form

$$\left(\frac{\partial}{\partial \mathcal{Z}^{AB}} + \hbar\,\mathcal{W}_{(A}\frac{\partial}{\partial \mathcal{Y}^{B)}}\right) F = 0\,. \tag{7.1}$$

Let us explain how they result from the nonstandard BRST operator construction. Using the rank r BRST-operator (4.12) for $r = 2$ one can write the current equations in the coordinate independent $SpH(2M)$ invariant form.

Setting $\nu_1 = \nu$ and $\nu_2 = -\nu$ and using notations $c_\pm = c_1 \pm c_2$ and $J_\pm = \frac{1}{2}(J_1 \pm J_2)$ we obtain

$$\begin{aligned}\mathbf{Q}_2\big|_{b=0} = {} & c^A{}_B(J^B{}_A + \delta^B{}_A) + c^{AB}(J_{AB} - 4\nu^{-1}J_{+A}J_{-B}) + c_{AB}J^{AB} + \\ & + c_+J_+ + c_-(J_- - \nu) + c_{+A}J_+{}^A + c_{-A}J_-{}^A.\end{aligned} \tag{7.2}$$

So, for a rank 2 field $F(\mathcal{A},\mathcal{C},X,y_\pm,w_\pm,u_\pm)$ independent of the Clifford elements $c_\pm$ and $b_\pm$, the condition

$$\mathbf{Q}_2 F = 0$$

implies

$$J_+ F = 0\,, \tag{7.3}$$

$$(J_- - \nu)F = 0\,, \tag{7.4}$$

$$(J^B{}_A + \delta^B{}_A)F = 0\,, \tag{7.5}$$

$$J^{AB} F = 0\,, \tag{7.6}$$

$$J_+{}^A F = 0\,, \tag{7.7}$$

$$J_-{}^A F = 0\,, \tag{7.8}$$

$$(J_{AB} - 4\nu^{-1} J_{-(B} J_{+A)})F = 0\,. \tag{7.9}$$

Substituting the expressions (5.2), (5.5) and (5.6) to (7.3)–(7.6) we obtain

$$\frac{\partial}{\partial u_+} F = 0, \tag{7.10}$$

$$\Big(\frac{\partial}{\partial u_-} - \frac{1}{2}\nu\Big) F = 0, \tag{7.11}$$

$$\Big(\mathcal{A}^C{}_A \frac{\partial}{\partial \mathcal{A}^C{}_B} - \delta^B_A\Big) F = 0, \tag{7.12}$$

$$2\frac{\partial}{\partial \mathcal{C}_{AB}} F = 0. \tag{7.13}$$

By virtue of (7.13) we can set $\mathcal{C} = 0$. So, using (7.10) we obtain from (7.7) and (7.8)

$$\Big(\frac{\partial}{\partial w_{+D}} + X^{DA}\frac{\partial}{\partial y_+{}^A} + \frac{1}{4}\nu\, Y_-{}^D\Big) F = 0, \tag{7.14}$$

$$\Big(\frac{\partial}{\partial w_{-A}} + X^{AB}\frac{\partial}{\partial y_-{}^B} + \frac{1}{4}\nu\, Y_+{}^A\Big) F = 0. \tag{7.15}$$

Since the equations (7.15), (7.14) reconstruct the dependence on $w_\pm$, we can set $w_\pm = 0$ in (7.9), whence by virtue of (7.13), (7.10), (5.5) and (5.6) we have

$$\Big(\frac{\partial}{\partial X^{AB}} + 2\nu^{-1}\,\frac{\partial}{\partial Y_-{}^{(B}}\,\frac{\partial}{\partial Y_+{}^{A)}}\Big) F = 0\,. \tag{7.16}$$

Setting

$$\frac{\partial}{\partial W_+} = Y_- \qquad W_+ = -\frac{\partial}{\partial Y_-}\,,$$

which substitution is analogous to the Fourier transform (1.17), we obtain from (7.16)

$$\Big(\frac{\partial}{\partial X^{AB}} - 2\nu^{-1}\, W_{+(B}\,\frac{\partial}{\partial Y_+{}^{A)}}\Big) F = 0\,. \tag{7.17}$$

Analogously for the complex version of the vector fields (5.2), (5.5) and (5.6) we obtain

$$\Big(\frac{\partial}{\partial \mathcal{Z}^{AB}} - 2\nu^{-1}\, \mathcal{W}_{+(B}\,\frac{\partial}{\partial \mathcal{Y}_+{}^{A)}}\Big) F = 0\,. \tag{7.18}$$

Up to notations and coefficients this gives the current equations (7.1).

8 Unfolded dynamics and twistors

In this section we touch very briefly some general aspects of the analogy between the unfolded dynamics approach and twistors.

Let M^d be a d dimensional manifold with coordinates x^n $(n = 0, 1, \dots d-1)$. By unfolded formulation of a linear or nonlinear system of differential equations in M^d we mean its equivalent reformulation in the first-order form

$$dW^\Phi(x) = G^\Phi(W(x))\,, \tag{8.1}$$

where $d = dx^n \dfrac{\partial}{\partial x^n}$ is the exterior differential on M^d, $W^\Phi(x)$ is a set of degree p_Φ-differential forms and $G^\Phi(W)$ is some degree $p_\Phi + 1$ function of the differential forms W^Φ

$$G^\Phi(W) = \sum_{n=1}^{\infty} f^\Phi{}_{\Omega_1 \dots \Omega_n} W^{\Omega_1} \wedge \dots \wedge W^{\Omega_n}\,,$$

where the coefficients $f^\Phi{}_{\Omega_1\dots\Omega_n}$ satisfy the (anti)symmetry condition $f^\Phi{}_{\Omega_1\dots\Omega_k\Omega_{k+1}\dots\Omega_n} = (-1)^{p_{\Omega_{k+1}} p_{\Omega_k}} f^\Phi{}_{\Omega_1\dots\Omega_{k+1}\Omega_k\dots\Omega_n}$ (extension to the supersymmetric case with an additional boson-fermion grading is straightforward) and G^Φ satisfies the condition

$$G^\Omega(W) \wedge \frac{\partial G^\Phi(W)}{\partial W^\Omega} = 0 \tag{8.2}$$

(the derivative $\frac{\partial}{\partial W^\Omega}$ is left) equivalent to the generalized Jacobi identity on the structure coefficients

$$\sum_{n=0}^{m} (n+1) f^\Lambda{}_{[\Phi_1\dots\Phi_{m-n+1}} f^\Phi{}_{\Lambda\Phi_{m-n}\dots\Phi_m\}} = 0\,, \tag{8.3}$$

where the brackets $[\,\}$ denote appropriate (anti)symmetrization of indices Φ_i. Given solution of (8.3) it defines a free differential algebra [46, 47].

This method of describing dynamical systems was originally proposed in [5, 6] where it was applied to the analysis of free and interacting massless gauge fields in four dimensional anti-de Sitter space. The name *unfolded formulation* was given somewhat later [7]. The method turned out to be very efficient and was further developed in a number of papers (see e.g. [8, 21, 31, 48] for recent discussions).

Let us note that, to some extent, unfolded formulation is analogous to the Cartan prolongation approach with the important difference however that it is extended to dynamical fields that are differential forms. This is crucial in several respects. In particular, in this approach geometry is described by differential forms via vielbein one form (ladder form) and Lorentz connection. The same time, differential forms are gauge fields analogous to vector potential in spin one Maxwell-Yang-Mills theory or vielbein in spin two gravity theory. The presence of gauge fields is crucial for interacting (i.e., nonlinear) theories. In addition, the exterior algebra formalism makes the unfolded equations coordinate independent and manifestly gauge invariant (in the latter case provided that the system is universal [8, 31]. Note that in this case the unfolded system amounts [8, 29] to some L_∞ algebra [49]).

JHEP12(2009)021

An important property of the unfolded dynamics is that, in the topologically trivial situation, degrees of freedom are concentrated in zero-forms $\omega_0^i(x_0)$ at any $x = x_0$. This is a consequence of the Poincare' lemma: the unfolded equations express all exterior derivatives in terms of the values of fields themselves modulo exact forms that can be gauged away by the gauge transformations. Locally, what is left is the "constant part" of the zero-forms.

This simple observation has a consequence that, to describe a system with an infinite number of degrees of freedom like a massless field, it is necessary to work with an infinite set of zero-forms that form an infinite dimensional module of the space-time symmetry $\mathfrak{g}$ ($\mathfrak{g} = \mathfrak{sp}(8)$, $\mathfrak{su}(2,2)$, $\mathfrak{iso}(1,3)$ etc). In fact, the module carried by zero-forms turns out to be dual (complex equivalent) to the space of single-particle states in the respective quantum field theory [4] which is the theory of massless fields of all spins in the case of most interest in this paper.

Usually, infinite sets of zero-forms are realized as a space of functions on some auxiliary space $\mathbf{T}$ *à la* (1.6). Particular examples are provided by the "generalized Weyl tensors" $C(Y|X)$ or $C(y,\bar{y}|x)$ discussed in section 1. In the gauge theory of higher-spin fields they describe gauge invariant field strengths built from gauge connection one-forms (for more detail see [21] and references therein).

$\mathbf{T}$ is an analogue of the twistor space in twistor theory. The most important difference with twistor theory is that, in the unfolded dynamics approach, the symmetry G is not necessarily geometrically realized in $\mathbf{T}$ (i.e., by vector fields at the infinitesimal level). Typically, $\mathbf{T}$ is realized as a Fock module where G acts via embedding into the Weyl algebra of oscillators. In particular, $G = \mathrm{Sp}(8)$ acts just this way in $\mathbf{T}$. This usually leads to the appearance of operators nonlinear in the annihilation operators (i.e., $\frac{\partial}{\partial Y^A}$ in our case) that act by higher-order differential operators hence driving us away from the usual twistor setup.

In general, two types of models may appear upon unfolding. The *effective* type is that where the forms W^Λ in the unfolded dynamics turn out to be unrestricted in the twistor space $\mathbf{T}$, or restricted by simple homogeneity conditions, eventually implying that $\mathbf{T}$ is a projective space as is the case if one considers a field of definite spin like in the standard twistor theory. The *ineffective* type is that where the 0-forms responsible for local degrees of freedom are not arbitrary in an appropriate space $\mathbf{T}$ being themselves described as solutions of differential conditions in $\mathbf{T}$ that may be as complicated as the original field equations.

In the effective case the unfolded field equations map arbitrary functions on $\mathbf{T}$ to solutions of the field equations on space-time $\mathbf{M}$, thus describing a Penrose transform. In the ineffective case, the unfolded field equations are still useful in many respects (in particular for introducing nonlinear field interactions in a coordinate-independent way [50]), but may be not particularly helpful for finding their explicit solutions.

There are two classes of models in higher-spin theory. The vector-like, that work in any dimension [50] are ineffective in the sense that the structure of zero-forms is fairly complicated requiring reductions and factorizations of ideals in certain noncommutative spaces. For example, in [8] it was shown that, in the vector-like approach, Einstein and Yang-Mills equations are unfolded in such a way that the corresponding zero-forms should themselves satisfy the Einstein and Yang-Mills equations in $\mathbf{T}$.

The spinor-like (or twistor-like) models are effective, operating with zero-forms valued in unrestricted spaces of spinor variables like Y^A. So far, these models have been formulated at the nonlinear level only in three and four space-time dimensions (see [9] and references therein). However, as has been argued in [4, 10, 16] models of this class are likely to allow extension to the Sp$(2M)$ case considered in this paper and, as a result, to higher dimensions including $d = 6, 10, 11$.

9 Conclusions

We hope that this paper sheds light on the relation between unfolded dynamics and twistor theory. The unfolded formulation is somewhat less restrictive, which means of course that some of the methods of twistor theory may not be directly extended to unfolded dynamics. For example, we have shown that the twistor-like description of Sp(8) invariant field equations for massless fields requires a nontrivial extension of the standard twistor approach. Interestingly enough, the current equation proposed in [22], which ensures the current conservation in Sp(8) invariant field theories, belongs to the normal twistor case of first order equations whose geometric meaning consists of the factorization of the correspondence space to the twistor space.

The parallelism between unfolded dynamics and twistor theory goes far enough. In particular, the unfolded dynamics approach effectively reformulates dynamical field equations in terms of $\mathfrak{g}$-modules where $\mathfrak{g}$ is a Lie algebra where one-forms take their values (for more detail see [8, 21, 31] and references therein). This definitely has a lot of similarity with the general approach of [2]. In particular, dynamical content of unfolded equations (independent fields, invariant differential operators, etc) is computed in terms of the so-called σ_- cohomology [30]. (For more detail see [4, 15, 31].) For example, the Sp(8) invariant equations (1.4) and (1.5) were derived by this method in [4]. We expect that σ_- cohomology should be related to the sheaf cohomology in twistor theory. We hope to analyze this and other questions on the interplay between unfolded dynamics and twistor theory in the future.

Another interesting problem for the future is to establish a formal correspondence of the proposed BRST construction with the unfolded equations which should in some sense be dual to each other. In particular, the unfolded formulation is also globally defined once left invariant forms on the corresponding group manifold $SpH(2M)$ are given. The latter however can be directly read of the Lie vector fields on $SpH(2M)$. This suggests that there should be a direct way to relate the two approaches. Hopefully, the BRST approach proposed in [42, 44] may be useful in this respect.

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JHEP12(2009)021

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JHEP12(2009)021

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JHEP12(2009)021